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A note on flux induced superpotentials in string theory

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Abstract: Non-vanishing fluxes in $\mathcal{M}$-theory and string theory compactifications induce a superpotential in the lower dimensional theory. Gukov has conjectured the explicit form of this superpotential. We check this conjecture for the heterotic string compactified on a Calabi-Yau three-fold as well as for warped $\mathcal{M}$-theory compactifications on Spin(7) holonomy manifolds, by performing a Kaluza-Klein reduction.

Keywords: M-Theory, Superstrings and Heterotic Strings.

Contents

1. Introduction

Compactifications of string theory and $\mathcal{M}$-theory with non-vanishing expectation values for tensor fields play a very special role, when trying to find a realistic string theory model that could describe our four-dimensional world. Especially interesting are the so called warped compactifications. Such compactifications were first discovered for the heterotic string in [1] and [2] and were later generalized to warped compactifications of $\mathcal{M}$-theory and $\mathcal{F}$-theory in [3, 4, 5]. In these compactifications tensor fields acquire non-vanishing expectation values, while leaving supersymmetry unbroken. The restrictions imposed by supersymmetry on the fluxes lead to constraints on the moduli fields of the theory, so that most of these fields will be stabilized. This has important phenomenological implications and will lead us one step closer to determining the coupling constants of the standard model out of string theory. The constraints imposed by supersymmetry for compactifications of $\mathcal{M}$-theory on a Calabi-Yau four-fold[1] were derived in [3]. These constraints tell us, that the internal component of the four-form F of $\mathcal{M}$-theory is of type $(2,2)$ and satisfies the primitivity condition

$$F \wedge J = 0\,. \tag{1.1}$$

It was shown in [9], that these conditions can be derived from two superpotentials, that describe the vacuum solutions in three dimensions[2]

$$W = \int_{Y_4} F \wedge \Omega\,, \tag{1.2}$$

[1]Compactifications of type-II theories on Calabi-Yau four-folds were considered in [6, 7, 8].

[2]In a nice paper by De Wolfe and Giddings [10] the effect of the warp factor on the superpotentials has been computed.

and

$$\hat{W} = \frac{1}{4}\int_{Y_4} F \wedge J \wedge J\,. \tag{1.3}$$

Here Ω describes the holomorphic four-form of the Calabi-Yau four-fold Y_4, while J describes the Kähler form. Strictly speaking (1.3) is not a superpotential, as it is not a holomorphic function of the Kähler moduli. Nevertheless, we will be using this terminology, as it has become standard in the literature. The above superpotentials and the corresponding scalar potential were derived from a Kaluza-Klein compactification of the $\mathcal{M}$-theory action in [11]. In supergravity theories with four supercharges the conditions for unbroken supersymmetry for compactifications to three-dimensional Minkowski space are [9]

$$W = D_\alpha W = 0\,, \tag{1.4}$$

and

$$\hat{W} = D_A \hat{W} = 0\,. \tag{1.5}$$

Here $\alpha = 1, \ldots, h^{13}$ describe the deformations of the complex structure and $A = 1, \ldots, h^{11}$ parametrize the deformations of the Kähler structure. In these formulas the derivatives are defined as $D_\alpha W = \partial_\alpha W + (\partial_\alpha K)W$ and $D_A \hat{W} = \partial_A \hat{W} - \frac{1}{2}(\partial_A K)\hat{W}$, where

$$K = -\log \int_{Y_4} \Omega \wedge \bar{\Omega} + \log \mathcal{V}\,, \tag{1.6}$$

is the three-dimensional Kähler potential. The constraints for a supersymmetric three-dimensional vacuum found in [3], then easily follow from the above conditions [9]. Let us briefly go through the argument. First, $W = 0$ implies $F_{4,0} = F_{0,4} = 0$. Second, the identity $D_\alpha W = \int \Phi_\alpha \wedge F$, where Φ_α is a basis of $(3,1)$-forms, gives together with $D_\alpha W = 0$ the condition $F_{1,3} = F_{3,1} = 0$. Third, the condition $D_A \hat{W} = 0$ implies together with $\hat{W} = 0$, that F is primitive. However, Calabi-Yau four-folds are not the whole story, as the resulting theory is three-dimensional and has an $\mathcal{N} = 2$ supersymmetry. Compactifications on other riemannian manifolds of exceptional holonomy are of special interest, as they allow us to obtain theories with less supersymmetry and in a different number of space-time dimensions. Recall, that there is a close connection between the theory of riemannian manifolds with reduced holonomy and the theory of calibrated geometry [12]. Calibrated geometry is the theory, which studies calibrated submanifolds, a special kind of minimal submanifolds of a riemannian manifold, which are defined using a closed form called the calibration. Riemannian manifolds with reduced holonomy usually come equipped with one or more natural calibrations. Based on this close relation to calibrated geometry and generalizing the result for the superpotential found in [9], Gukov made a conjecture about the form of the superpotential appearing in string theory compactifications with non-vanishing Ramond-Ramond fluxes on a manifold X of reduced holonomy [13]

$$W = \sum \int_X (\text{calibrations}) \wedge (\text{Fluxes})\,. \tag{1.7}$$

In this formula we sum over all possible combinations of fluxes and calibrations. This conjecture has been checked by computing the scalar potential from a Kaluza-Klein reduction of the action for a certain type of theories. This in turn, determines the superpotential. For the type-IIB theory these potentials have been computed in [14] and [15]. The superpotentials for type-IIA compactifications on Calabi-Yau four-folds were derived in [6, 7, 8], while the scalar potential for $\mathcal{M}$-theory on G_2-holonomy manifolds has been computed in [16]. Our goal will be to compute the superpotential for two different theories. Rather than computing the scalar potential and from there obtain the superpotential, we shall compute the superpotential directly by a Kaluza-Klein compactification of the gravitino supersymmetry transformation. We shall illustrate the idea in section 2. In section 3 we will compute the superpotential for the heterotic string compactified on a Calabi-Yau three-fold. It is well known, that for a conventional compactification of the heterotic string on a Calabi-Yau three-fold, i.e. without taking warp factors into account, turning on an expectation value for the heterotic three-form will induce a superpotential, which breaks supersymmetry without generating a cosmological constant [17]. In the context of Gukov's conjecture [13], it was argued in [18], that this superpotential can be written as in (1.7), generalizing the original proposal [13] to fluxes of Neveu-Schwarz type.[3] We shall check this conjecture by computing the superpotential explicitly from a Kaluza-Klein reduction of the gravitino supersymmetry transformation. In section 4 we shall apply a similar approach to compute the superpotential for $\mathcal{M}$-theory compactifications on a Spin(7) holonomy manifold. In the appendix we will review some relevant aspects of Spin(7) holonomy manifolds.

2. Gauge invariant supergravity lagrangian

In order to derive the superpotential for the four-dimensional heterotic string, it is easiest to compactify the gravitino supersymmetry transformation law. Recall, that the most general gauge invariant $\mathcal{N}=1$, $D=4$ supergravity action can be described in terms of three functions (see e.g. [19, 20]). These are the superpotential W, the Kähler potential K and a holomorphic function H_{ab}, which plays the role of the gauge coupling. In the following we will take $H_{ab}=\delta_{ab}$. The theory is formulated in terms of massless chiral multiplets, containing a complex scalar ϕ and a Weyl spinor ψ and massless vector multiplets, containing the field A^a_μ with field strength $F^a_{\mu\nu}$ and a Weyl spinor λ^a. We shall be adding a real auxiliary field D^a to the vector multiplets. The bosonic part of the lagrangian takes the following form[4]

$$\mathcal{L} = -\frac{1}{2}R - K_{\bar{i}j}D_\mu\phi^{i*}D^\mu\phi^j - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - V(\phi,\phi^*)\,. \tag{2.1}$$

Here $V(\phi,\phi^*)$ describes the scalar potential given by

$$V(\phi,\phi^*) = \exp(K)\left(K^{\bar{i}j}W^*_i W_j - 3W^*W\right) + \frac{1}{2}D^a D_a\,. \tag{2.2}$$

In this formula $K^{\bar{i}j}$ is the inverse matrix to $K_{\bar{i}j}$,

$$K_{\bar{i}j} = \frac{\partial^2 K(\phi,\phi^*)}{\partial\phi^{i*}\partial\phi^j}\,, \tag{2.3}$$

[3] See also [17] for an earlier discussion of the superpotential.

[4] We will be following the conventions of [20].

and $W_i = \partial_i W + \partial_i K\ W$. The complete lagrangian is invariant under $\mathcal{N}=1$ supersymmetry. The relevant part of the supersymmetry transformations takes the form

$$\begin{aligned} \delta\lambda^a &= F^a_{\mu\nu}\ \sigma^{\mu\nu}\epsilon - iD^a\ \epsilon\,, \\ \delta\psi_\mu &= 2\nabla_\mu\epsilon + i\ e^{K/2}\ \gamma_\mu\epsilon^* W\,. \end{aligned} \tag{2.4}$$

Here λ^a and ψ_μ are positive chirality Weyl spinors, describing the gluino and gravitino respectively, ϵ is a four-dimensional Weyl spinor of positive chirality, while ϵ^* is the complex conjugate spinor with negative chirality. If the space-time is flat, the complete supersymmetry transformations tell us that supersymmetry demands (see [20])

$$W_i = D^a = W = 0\,. \tag{2.5}$$

In the next section we will use the above supersymmetry transformations to determine the superpotential and D-term for the heterotic string compactified on a Calabi-Yau three-fold.

3. Superpotential for the heterotic string on a Calabi-Yau three-fold

It has been known for a long time, that gluino condensation triggers spontaneous supersymmetry breaking in the heterotic string compactified on a Calabi-Yau three-fold Y_3 (with no warp factors) without producing a vacuum energy [17]. In this process the Neveu-Schwarz three-form H of the heterotic string acquires a vacuum expectation value proportional to the holomorphic three-form Ω of the Calabi-Yau three-fold. It was shown in [17], that this generates a superpotential, which will break supersymmetry completely. In a more recent context[5] it was argued in [18], that the superpotential which is induced by such a non-vanishing H-field

$$W = \int_{Y_3} H\wedge\Omega\,, \tag{3.1}$$

extends the conjecture (1.7) to superpotentials with non-vanishing fluxes of Neveu-Schwarz type. The argument, which motivated the above formula, was based on the identification of BPS domain walls with branes wrapped over supersymmetric cycles. More concretely, the BPS domain wall of the $\mathcal{N}=1$, $D=4$ theory originates from the heterotic five-brane wrapping a special lagrangian submanifold of Y_3. This is because the five-brane is a source for the Neveu-Schwarz three-form field strength H. Here we would like to compute the form of the superpotential and the form of the D-term appearing in (2.4) in this particular model by a direct Kaluza-Klein reduction of the gravitino and gluino supersymmetry transformation respectively. We then would like to compare the result with formula (3.1). Recall that the ten-dimensional $\mathcal{N}=1$ supergravity multiplet contains a metric g_{MN}, a spin-3/2 field Ψ_M, a two-form potential B_{MN}, a spin-1/2 field λ and a scalar field ϕ. The super-Yang-Mills multiplet contains the Yang-Mills field F^a_{MN} and a

[5] For an earlier discussion of the form of the superpotential see [17].

spin-1/2 field χ^a, the so called gluino. The relevant part of the $\mathcal{N} = 1$ supersymmetry transformations in the ten-dimensonal string frame take the form

$$\begin{aligned}\delta\Psi_\mu &= \nabla_\mu \eta + \frac{1}{48}(\gamma_\mu\gamma_5 \otimes \gamma^{abc} H_{abc})\,\eta\,,\\ \delta\chi^\alpha &= -\frac{1}{4} F^\alpha_{ab}\gamma^{ab}\eta\,.\end{aligned} \tag{3.2}$$

Here μ describes the coordinates of the four-dimensional Minkowski space, and $a, b, \ldots$ describe the six-dimensional internal indices, while α describes the gauge index.

We consider a Majorana representation for ten-dimensional Dirac matrices with Γ_M real and hermitian, apart from Γ_0 which is real and antihermitian. The matrices Γ_M can be represented as tensor products of γ_μ, the matrices of the external space, with γ_m, the matrices of the internal space

$$\begin{aligned}\Gamma_\mu &= \gamma_\mu \otimes 1\,,\\ \Gamma_m &= \gamma_5 \otimes \gamma_m\,,\end{aligned} \tag{3.3}$$

with

$$\gamma_5 = \frac{i}{4!}\,\epsilon_{\mu\nu\rho\sigma}\,\gamma^{\mu\nu\rho\sigma}\,. \tag{3.4}$$

We can also introduce the matrix

$$\gamma = \frac{i}{6!}\sqrt{g_{(6)}}\,\epsilon_{mnpqrs}\,\gamma^{mnpqrs}\,, \tag{3.5}$$

which determines the chirality in the internal space. Here $g_{(6)}$ represents the determinant of the internal metric. Thus γ_μ are real and hermitian apart from γ_0 which is real and antihermitian and γ_m are imaginary and hermitian as are γ_5 and γ. The relation between Γ, the matrix which determines the chirality in ten-dimensions, γ_5 and γ is

$$\Gamma = -\,\gamma_5 \otimes \gamma\,. \tag{3.6}$$

Consider η a ten-dimensional Majorana-Weyl spinor of positive chirality. In order to compactify transformations (3.2) to four dimensions, we decompose this ten-dimensional spinor in terms of the covariantly constant spinors of the internal manifold:

$$\eta = \epsilon^* \otimes \xi_+ + \epsilon \otimes \xi_-\,, \tag{3.7}$$

where ξ_+ and $\xi_- = (\xi_+)^*$ are six-dimensional Weyl spinors with positive and negative chirality respectively and ϵ is a four-dimensional Weyl spinor of positive chirality, whose complex conjugate is ϵ^*. Similarly, we decompose the ten-dimensional gravitino as:

$$\Psi_\mu = \psi^*_\mu \otimes \xi_+ + \psi_\mu \otimes \xi_-\,, \tag{3.8}$$

where ψ_μ is a four-dimensional Weyl spinor of positive chirality, that represents the four-dimensional gravitino.

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In complex coordinates the gravitino supersymmetry transformation takes the form

$$\delta\Psi_\mu = \nabla_\mu \eta + \frac{1}{48}\left(\gamma_\mu\gamma_5 \otimes \left(\gamma_{mnp}H^{mnp} + \gamma_{\bar m\bar n\bar p}H^{\bar m\bar n\bar p}\right)\right)\ \eta + \\ + \frac{1}{48}\left(\gamma_\mu\gamma_5 \otimes\ \left(\gamma_{mn\bar p}H^{mn\bar p} + \gamma_{m\bar n\bar p}H^{m\bar n\bar p}\right)\right)\ \eta\,. \tag{3.9}$$

To evaluate the resulting expressions we use the identities (see e.g. [21] or [22])

$$\gamma_{\bar m}\xi_+ = 0\,, \tag{3.10}$$

$$\gamma_{mnp}\xi_+ = \|\xi_+\|^{-2}\ \Omega_{mnp}\ \xi_-\,,$$
$$\gamma_{mn\bar p}\xi_+ = 2i\ \gamma_{[m}J_{n]\bar p}\ \xi_+\,,$$
$$\gamma_{m\bar n\bar p}\xi_+ = \gamma_{\bar m\bar n\bar p}\xi_+ = 0\,. \tag{3.11}$$

We now decompose our ten-dimensional spinors as in (3.7) and (3.8) and make use of formulas (3.10) and (3.11). Multiplying the resulting expression from the left with $\xi_-^\dagger = \xi_+^T$, we obtain the transformation:

$$\delta\psi_\mu = \nabla_\mu\epsilon - \frac{1}{48}\gamma_\mu\epsilon^*\,\|\xi_+\|^{-2}\ H_{\bar m\bar n\bar p}\,\Omega^{\bar m\bar n\bar p}\,. \tag{3.12}$$

After integration over the internal manifold we obtain:

$$\delta\psi_\mu = \nabla_\mu\epsilon + i\gamma_\mu\epsilon^*\,\|\xi_+\|^{-2}\,e^{K_2}\ \int_{Y_3} H\wedge\Omega\,. \tag{3.13}$$

where we have used that:

$$V = \frac{1}{48}\int_{Y_3} J\wedge J\wedge J = \frac{1}{64}\,e^{-K_2}\,, \tag{3.14}$$

with V being the volume of the internal Calabi-Yau manifold. If we choose

$$\|\xi_+\|^{-2} = e^{K/2-K_2} \tag{3.15}$$

and rescale the fields:

$$\psi \to \frac{\psi}{2}\,,$$
$$H \to \frac{H}{2}\,, \tag{3.16}$$

then we obtain the four-dimensional supersymmetry transformation for gravitino,

$$\delta\psi_\mu = 2\nabla_\mu\epsilon + i\gamma_\mu\epsilon^*\,e^{K/2}\ \int_{Y_3} H\wedge\Omega\,. \tag{3.17}$$

In the above formulas $K = K_1 + K_2$ is the total Kähler potential, where K_1 is the Kähler potential for complex structure deformations

$$K_1 = -\log\left(i\int_{Y_3}\Omega\wedge\bar\Omega\right)\,, \tag{3.18}$$

and K_2 is the Kähler potential for the Kähler deformations

$$K_2 = -\log\left(\frac{4}{3}\int_{Y_3} J\wedge J\wedge J\right). \tag{3.19}$$

Comparing this result with (2.4) we find the superpotential

$$W = \int_{Y_3} H\wedge\Omega\,, \tag{3.20}$$

as promised.

Let us now consider the gluino supersymmetry transformations in (3.2). If we again decompose the gluino as in (3.8) and the spinor η as in (3.7), we obtain after comparing with (2.4) the form of the four-dimensional D-term up to a multiplicative constant

$$D^a = F^a_{m\bar{n}} J^{m\bar{n}}\,. \tag{3.21}$$

Here we have used

$$J_{m\bar{n}} = -i\xi_+^\dagger \gamma_{m\bar{n}}\xi_+\,, \tag{3.22}$$

while the expectation value for the other index contractions appearing in the four-dimensional gluino supersymmetry transformation vanish. As we have mentioned in the previous section, supersymmetry demands $D^{(a)} = 0$, which in this case gives the, well known, Donaldson-Uhlenbeck-Yau equation

$$J^{m\bar{n}} F^a_{m\bar{n}} = 0\,. \tag{3.23}$$

The fact that the Donaldson-Uhlenbeck-Yau equation originates from a D-term constraint was first discussed in [23]. Furthermore, supersymmetry demands

$$W_i = 0\,, \tag{3.24}$$

where we are using again the notation $W_i = \partial_i W + \partial_i K_1\ W$, where K_1 is the Kähler potential for complex structure deformations and is given in (3.18). It is straightforward to evaluate this constraint to obtain $W_i = \int_{Y_3}\phi_i\wedge\Omega = 0$, where ϕ_i is a complete set of $(2,1)$ forms [24]. This implies that H is of type $(0,3)$. However in this case

$$W \neq 0\,, \tag{3.25}$$

and we therefore see, that no supersymmetric solutions can be found. It is expected, that this situation changes, if we consider instead a 'warped' compactification of the heterotic string [1, 2]. The resulting background is in this case a complex manifold with non-vanishing torsion.[6] It is expected that supersymmetric ground states can be found in this case. In [26] we have already computed the form of this superpotential and checked, that it takes the same form as (3.20), but now the complete Chern-Simons terms of H have to be taken into account

$$H = dB + \omega_L - \omega_{YM}\,. \tag{3.26}$$

In the supergravity approximation, that we have been using, it is only possible to take the Chern-Simons term of the gauge field into account ω_{YM}, as the Chern-Simons term ω_L coming from the spin connection is a higher order effect. We shall report on more details

[6] Manifolds with non-vanishing torsion have also been discussed some time ago in e.g. [25]. As opposed to the previous reference, the manifolds we shall be interested in have a torsion that is not closed.

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elsewhere [26]. At the same time, there is not much known about the mathematical properties of these background manifolds, as many of the theorems on complex manifolds do not generalize easily to the case of a manifold with non-vanishing torsion. Very generally, not many examples of such manifolds with torsion are known. A few of them have been constructed in [5]. A rather interesting concrete example of such a compactification of the heterotic string on a manifold with torsion has recently appeared [27]. Here it was shown, that the supersymmetry constraints derived in [1] are satisfied for this particular background manifold. Very generally, it would be interesting to understand the mathematical properties of these manifolds with torsion. Work in this direction is in progress [26].

4. The superpotential for Spin(7) holonomy manifolds

In this section we would like to consider warped compactifications of $\mathcal{M}$-theory on a Spin(7) holonomy manifold X. The resulting action has an $\mathcal{N}=1$ supersymmetry in three dimensions. These theories are closely related to four-dimensional counterparts with completely broken supersymmetry. This is because the dimensional reduction of the minimal $\mathcal{N}=1$, $D=4$ supersymmetric theory to three dimensions would lead to an $\mathcal{N}=2$ supersymmetric theory. Models with $\mathcal{N}=1$ supersymmetry in three dimensions are interesting in connection to the solution of the cosmological constant problem along the lines proposed by Witten in [28] and [29]. The basic idea of this proposal is, that in three dimensions supersymmetry can ensure the vanishing of the cosmological constant, without implying the unwanted Bose-Fermi degeneracy. However, this mechanism does not explain, why the cosmological constant of our four-dimensional world is so small, unless there is a duality between a three-dimensional supersymmetric theory and a four-dimensional non-supersymmetric theory of the type, that we are discussing. So, $\mathcal{M}$-theory compactifications on Spin(7) holonomy manifolds allow us to address the cosmological constant problem from the three-dimensional perspective.

The mathematical aspects of Spin(7) holonomy manifolds have extensively been studied in the literature (see e.g. [30], where examples of compact manifolds have been discussed). Recently, there have also been constructed examples of such manifolds, which are not compact [31] and [32].

Much less is known about the form of the most general three-dimensional action with $\mathcal{N}=1$ supersymmetry, describing the coupling of gauge fields to supergravity. Some aspects, such as the field content of compactifications of $\mathcal{M}$-theory on Spin(7) holonomy manifolds have been studied in [33]. However, the complete form of the low energy effective action is not known at this point and work in this direction is in progress [34]. Very generally, it is known that the manifold of scalars is in this case riemannian instead of Kähler. It is expected, that the metric on this riemannian manifold can be determined in terms of a potential function[7]

$$\mathcal{K} = -\log \int_X \Omega \wedge \Omega \,, \tag{4.1}$$

[7] We thank G. Papadopoulos for discussions on this.

where Ω describes the Cayley calibration of the Spin(7) holonomy manifold. This is a closed and self-dual four-form $\Omega = *\Omega$. Furthermore, we expect that the three-dimensional action includes a superpotential, whose concrete form has been conjectured in [13]

$$W = \int_X \Omega \wedge F \,. \tag{4.2}$$

The constraints imposed by supersymmetry on these compactifications were derived in [35] and [36]. In [37] it was shown, that these constraints can be derived from the superpotential (4.2). Here we will check more directly, that the superpotential is given by (4.2), by performing a Kaluza-Klein reduction of the gravitino supersymmetry transformation law, along the lines of the previous section. The bosonic part for eleven-dimensional supergravity lagrangian contains a three-form C with field strength F and the dual seven-form $*F$, as well as the space-time metric g_{MN}

$$\mathcal{L} = \frac{1}{2}\int d^{11}x\sqrt{-g}\left(R - \frac{1}{2}F\wedge *F - \frac{1}{6}C\wedge F\wedge F\right). \tag{4.3}$$

Here we have set the gravitational constant equal to one. The supersymmetry transformation of the gravitino Ψ_M takes the form

$$\delta\Psi_M = \nabla_M\eta - \frac{1}{288}\left(\Gamma_M{}^{PQRS} - 8\delta_M^P\Gamma^{QRS}\right)F_{PQRS}\eta\,, \tag{4.4}$$

where capital letters denote eleven-dimensional indices and η is an eleven-dimensional anticommuting Majorana spinor. In order to compactify this theory on a Spin(7) holonomy manifold, we will make the following ansatz for the metric

$$g_{MN}(x,y) = \Delta^{-1}(y)\begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y)\end{pmatrix}, \tag{4.5}$$

where x are the coordinates of the external space labelled by the indices $\mu, \nu, \ldots$ and y are the coordinates of the internal manifold labelled by $m, n, \ldots$, while $\Delta = \Delta(y)$ is the warp factor. The eleven-dimensional spinor η is decomposed as

$$\eta = \epsilon \otimes \xi\,, \tag{4.6}$$

where ϵ is a three-dimensional anticommuting Majorana spinor and ξ is an eight-dimensional Majorana-Weyl spinor. Furthermore, we will make the following decomposition of the gamma matrices

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \otimes \gamma_9\,, \\ \Gamma_m &= 1 \otimes \gamma_m\,, \end{aligned} \tag{4.7}$$

where γ_μ and γ_m are the gamma matrices of the external and internal space respectively. We choose the matrices γ_m to be real and antisymmetric. γ_9 is the eight-dimensional chirality operator, which anti-commutes with all the γ_m's. In compactifications with maximally

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symmetric three-dimensional space-time the non-vanishing components of the four-form field strength F are

$$
\begin{aligned}
&F_{mnpq} \quad \text{arbitrary} \\
&F_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho} f_m \,,
\end{aligned}
\tag{4.8}
$$

where $\epsilon_{\mu\nu\rho}$ is the Levi-Civita tensor of the three-dimensional external space. The external component of the gravitino supersymmetry transformation is then given by the following expression [35]

$$
\begin{aligned}
\delta\Psi_\mu = {} & \nabla_\mu \eta - \frac{1}{288}\Delta^{3/2}(\gamma_\mu \otimes \gamma^{mnpq})F_{mnpq}\eta + \\
& + \frac{1}{6}\Delta^{3/2}(\gamma_\mu \otimes \gamma^m) f_m \eta + \frac{1}{4}\partial_n(\log\Delta)(\gamma_\mu \otimes \gamma^n)\eta \,,
\end{aligned}
\tag{4.9}
$$

where we have used a positive chirality eigenstate $\gamma_9\xi = \xi$. Considering negative chirality spinors corresponds to an eight-manifold with a reversed orientation [38]. We can decompose the eleven-dimensional gravitino as

$$
\Psi_\mu = \psi_\mu^{(3)} \otimes \xi \,,
\tag{4.10}
$$

where $\psi_\mu^{(3)}$ is the three-dimensional gravitino. After inserting (4.6) and (4.10) in (4.9), we multiply both sides of this equation from the left with the transposed spinor ξ^T. To evaluate the resulting expression we notice, that on these eight-manifolds it is possible to construct different types of p-forms in terms of the eight-dimensional spinor ξ as

$$
\omega_{a_1\ldots a_p} = \xi^T \gamma_{a_1\ldots a_p} \xi \,.
\tag{4.11}
$$

Since ξ is Majorana-Weyl, (4.11) is non-zero only for $p = 0, 4$ or 8 (see [39]). By this argument we notice, that the expectation values of the last two terms appearing in (4.9) vanish, as they contain only one internal gamma matrix. The

$$
\text{Spin}(7)
$$

calibration is given by the closed self-dual 4-form [36, 35]

$$
\Omega_{mnpq} = \xi^T \gamma_{mnpq} \xi \,.
\tag{4.12}
$$

We have to be a little more careful though, as the previous form is defined in terms of the covariantly constant spinor $\tilde{\xi} = \Delta^{1/4}\xi$ and gamma matrices that are rescaled by the warp factor in [35]. However, to the order we are working out the supersymmetry transformation, the warp factor can be taken to be constant. We therefore obtain from (4.9)

$$
\delta\psi_\mu^{(3)} = \nabla_\mu \epsilon + \gamma_\mu \epsilon \int_X F \wedge \Omega \,,
\tag{4.13}
$$

where we have again dropped a multiplicative constant in front of the second term on the right hand side. Luckily from [40] it is known, that the gravitino supersymmetry transformation in three dimensions contains a term of a similar form as in the four-dimensional case (2.4) but now formulated in terms of the three-dimensional Majorana spinor. We can then read off the form of the superpotential

$$W = \int_X F \wedge \Omega \,, \tag{4.14}$$

which is what we wanted to show.

Using this superpotential, it was shown in [37], that the $\mathcal{N} = 1$ supersymmetric vacua in three-dimensional Minkowski space found in [35] can be derived from the equations

$$W = W_i = 0\,. \tag{4.15}$$

Here W_i indicates the derivative of W with respect to the scalar fields, that come from the metric deformations of the Spin(7) holonomy manifold. From now on we will restrict our analysis to compact Spin(7) holonomy manifolds, even though the analysis of [35] was not restricted to that case. First we notice, that if X is an eight-manifold with Spin(7) holonomy, then the internal component of the four-form F can, in general, belong to the following cohomologies

$$H^4(X,\mathbb{R}) = H^4_{1+}(X,\mathbb{R}) \oplus H^4_{27+}(X,\mathbb{R}) \oplus H^4_{35^-}(X,\mathbb{R})\,. \tag{4.16}$$

The label "$\pm$" indicates self-dual and anti-self-dual four-forms respectively and the subindex indicates the representation. The Cayley calibration Ω belongs to the cohomology $H^4_{1+}(X,\mathbb{R})$. Getting back to the equation (4.15) we notice, that the condition $W = 0$ implies $F_{1+} = 0$, which is equivalent to equation (24) of [35]. According to [37] and [30], W_i generates the $H^4_{35^-}(X,\mathbb{R})$ cohomology, so that $W_i = 0$ implies $F_{35^-} = 0$. This is equation (21) of [35]. There are no more constraints on the fluxes for a compact manifold, as in this case the cohomology group $H^4_{7+}(X,\mathbb{R})$, vanishes. To summarize, supersymmetry demands that the flux on a compact Spin(7) holonomy manifold takes the form

$$F = F_{27+}\,. \tag{4.17}$$

We can now extend the arguments of [35] and [37], to check, if it is possible to find non-supersymmetric vacua with a vanishing three-dimensional cosmological constant as in [15, 11, 41, 42] and [8]. A not supersymmetric solution to the equations of motion will satisfy the condition $W_i = 0$ but W will be non-vanishing. This means that for compactifications on $\mathcal{M}$-theory on Spin(7) holonomy manifolds the equations of motion will be satisfied, if the internal component of the four-form field strength takes the form

$$F = F_{1+} \oplus F_{27+}\,. \tag{4.18}$$

Supersymmetry demands the additional constraint $W = 0$. As we had already seen, this means that the first term on the right hand side of the above expression vanishes. Therefore, in this type of compactifications it is possible to find a solution to the equations of motion in three-dimensional Minkowski space, that breaks supersymmetry, by turning on the form F_{1+}, without generating a cosmological constant. Such an interesting scenario with a vanishing cosmological constant and broken supersymmetry has already appeared in a number of different contexts [15, 11, 41, 42] and [8]. However, in contrast to the superpotentials appearing in the previous references, it is expected that the superpotential (4.14) receives perturbative and non-perturbative quantum corrections. For an analysis of some aspects of these corrections see [37]. This completes our discussion about $\mathcal{M}$-theory compactifications on Spin(7) holonomy manifolds.

5. Conclusions and outlook

In this paper we have checked the conjecture (1.7) made in [13] regarding the form of the superpotential, which is induced when non-trivial fluxes are turned on in different types of string theory and $\mathcal{M}$-theory compactifications. We do so, by performing a Kaluza-Klein reduction of the gravitino supersymmetry transformation. As it is well known from [17], a compactification of the heterotic string on a Calabi-Yau three-fold leads to a superpotential, which breaks supersymmetry completely. We have checked, that this superpotential can be written in the form (3.20), which extends the conjecture made in [13] to fluxes of Neveu-Schwarz type [18].

It is interesting, that the superpotential for warped compactifications of the heterotic string is given by (3.20), where the heterotic H field now includes the Chern-Simons terms [26]. For these compactifications, the internal manifold is then, in general, non-Kähler and has a non-vanishing torsion. Many theorems on complex manifolds do not easily generalize to manifolds with non-vanishing torsion. Therefore, these string theory compactifications are far from being well understood. These issues are presently under investigation and we will report on this in a near future [26].

In the second part of the paper we have considered warped compactifications of $\mathcal{M}$-theory on Spin(7) holonomy manifolds. These compactifications have been previously analyzed in [35, 36] and [37]. Not much is known about the form of the low energy effective action of the resulting three-dimensional $\mathcal{N} = 1$ theory and work in this direction is in progress [34]. Luckily it is known, that the gravitino supersymmetry transformation contains a similar term as in the four-dimensional case [40], so that we could verify the conjecture of [13] by a direct calculation of the superpotential. Using this superpotential, we have shown the existence of solutions to the three-dimensional equations of motion, which break supersymmetry and have a vanishing three-dimensional cosmological constant. Such an interesting scenario has recently appeared many times in the literature.

Contrary to the superpotential appearing in compactifications of $\mathcal{M}$-theory on Calabi-Yau four-folds, it is known that this $\mathcal{N} = 1$ superpotential receives perturbative and non-perturbative quantum corrections [37]. It would be nice to compute these corrections along the lines of [42].

Finally, it would be interesting to derive the constraints (4.18) from the direct analysis of the equations of motion, as in [41]. This might be a little bit more complicated as Spin(7) holonomy manifolds are, in general non-Kähler.

Note added. The superpotential for compactifications of the heterotic string on non-Kähler complex six-dimensional manifolds has now been computed in [42].

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A. Review of Spin(7) holonomy manifolds

This appendix contains a brief review of some of the relevant aspects of Spin(7) holonomy manifolds. A very nice and complete discussion can be found in [30]. On an riemannian manifold X of dimension n, the spin connection ω is, in general, an SO(n) gauge field. If we parallel transport a spinor ψ around a closed path γ, the spinor comes back as $U\psi$, where $U = P \exp \int_\gamma \omega \cdot dx$ is the path ordered exponential of ω around the curve γ.

A compactification of $\mathcal{M}$-theory (or string theory) on X preserves some amount of supersymmetry, if X admits one (or more) covariantly constant spinors. Such spinors return upon parallel transport to its original value, i.e. they satisfy $U\psi = \psi$. The holonomy of the manifold is then a subgroup of SO(n). A Spin(7) holonomy manifold is an eight-dimensional manifold, for which one such spinor exists. Therefore, if we compactify $\mathcal{M}$-theory on these manifolds we obtain an $\mathcal{N} = 1$ theory in three dimensions. Spin(7) is a subgroup of $GL(8, \mathbb{R})$ defined as follows. Introduce on $\mathbb{R}^8$ the coordinates $(x_1, \ldots, x_8)$ and the four-form $dx_{ijkl} = dx_i \wedge dx_j \wedge dx_k \wedge dx_l$. We can define a four-form Ω on $\mathbb{R}^8$ by

$$\begin{aligned} \Omega = \; & dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} - dx_{2358} - \\ & - dx_{2367} - dx_{2457} + dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678} \,. \end{aligned} \tag{A.1}$$

The subgroup of $GL(8, \mathbb{R})$ preserving Ω is the holonomy group Spin(7). It is a compact, simply connected, semisimple, twenty-one-dimensional Lie group, which is isomorphic to the double cover of SO(7). The form Ω is self-dual, i.e. it satisfies $\Omega = *\Omega$, where $*$ is the Hodge star of $\mathbb{R}^8$. Many of the mathematical properties of Spin(7) holonomy manifolds are discussed in detail in [30]. Let us here only mention that these manifolds are Ricci flat but are, in general, not Kähler. The cohomology of a compact Spin(7) holonomy manifold can be decomposed into the following representations of Spin(7)

$$\begin{aligned} H^0(X, \mathbb{R}) &= \mathbb{R} \\ H^1(X, \mathbb{R}) &= 0 \\ H^2(X, \mathbb{R}) &= H^2_{\mathbf{21}}(X, \mathbb{R}) \\ H^3(X, \mathbb{R}) &= H^3_{\mathbf{48}}(X, \mathbb{R}) \\ H^4(X, \mathbb{R}) &= H^4_{\mathbf{1}^+}(X, \mathbb{R}) \oplus H^4_{\mathbf{27}^+}(X, \mathbb{R}) \oplus H^4_{\mathbf{35}^-}(X, \mathbb{R}) \\ H^5(X, \mathbb{R}) &= H^5_{\mathbf{48}}(X, \mathbb{R}) \\ H^6(X, \mathbb{R}) &= H^6_{\mathbf{21}}(X, \mathbb{R}) \\ H^7(X, \mathbb{R}) &= 0 \\ H^8(X, \mathbb{R}) &= \mathbb{R} \,. \end{aligned} \tag{A.2}$$

The label "$\pm$" indicates self-dual and anti-self-dual four-forms respectively and the subindex indicates the representation. The Cayley calibration Ω belongs to the cohomology $H^4_{1+}(X,\mathbb{R})$. In this decomposition one has to take into account, that for a compact Spin(7) holonomy manifold $H^4_{7+}(X,\mathbb{R})=0$.

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